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Habit Formation and the Evolution of Social Communication Networks*

Edward Droste‡

Abstract

This paper analyzes a finite normal-form game, where the players' pure strategies consist of establishing links with the other players. The payoffs result from social communication between the players. We identify a social communication network with a mixed-strategy profile and we model the evolution of networks using a stochastic process of habit formation in discrete time. Börgers and Sarin [5] show that at every finite period of time, assuming frequent communication and slow adjustment, the stochastic process can be approximated by the continuous-time replicator dynamics. Using this result we are able to specify an asymptotically stable set of social communication networks which consists of Nash equilibria of the corresponding game. Furthermore, in the 3-player case this set reduces to the set consisting of the two efficient 'cyclic' social communication networks.

Journal of Economic Literature Classification Numbers: C72, C73.

KEYWORDS: Stochastic learning theory, game theory, network formation, social communication, replicator dynamics.

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1 Introduction

Human interactions in an economic context often have a social character. For example, one could think of talking with friends about job openings, working together with colleagues, and asking acquaintances about business opportunities. Recurring patterns of such interaction bind individuals together into social networks. Social networks as means to communicate information have a significant influence on the outcome of a variety of economic relationships. Ellison and Fudenberg [10] find that the structure of a communication process affects the diffusion of new products and technologies. According to Bolton and Dewatripont [3] and Radner [16] a firm's efficiency depends on the organizational structure through which information is shared among the employees. Information exchange networks influence stock market volatility as shown by Baker and Iyer [1].

In this paper we propose a probabilistic choice approach to social communication networks. Theories of probabilistic choice state that agents behave in a way that tends toward choosing better alternatives with higher probability than inferior ones. Alternatively, the algebraic approach (cf. Luce [13]), which is conventional in economics, assumes that choice probabilities are always zero or one and that the observed choices tell us which it actually is. The question which approach is most appropriate seems unanswerable. Empirical data on intransitivities of choices and inconsistencies when the same choices are offered several times suggests that a probabilistic model is preferable. The data is, however, far from conclusive as both phenomena can also be explained by an algebraic theory provided that preferences are allowed to change over time.

The literature offers two explanations for the apparently stochastic choices made by agents. A first explanation states that agents have some randomly drawn evaluations with respect to which they maximize utility. McKelvey and Palfrey [15] and McFadden [14] follow this way of reasoning by arguing that an agent's perception of the utility of the alternatives is subject to noise and that utility functions of agents are randomly drawn from some specific family, respectively.

A second explanation takes the view that agents are not utility maximizers but instead choose randomly in a fashion that is influenced by a subconscious utility function. If an agent's subconscious utility function specifies that one alternative is preferred to another, it will be chosen with higher probability. The decisions of an agent therefore reflect a rough recognition of his deterministic preference ordering which is buried rather deeply in his unconscious mind. This bounded rationality interpretation was pioneered by Chen, Friedman, and Thisse [7] and is also pursued in this paper.

In the finite normal-form game analyzed in this paper, a pure strategy of a player consist of forming a link with another player. A graph, with its nodes representing the players and its edges capturing the links, can therefore be associated with an outcome of the game. The payoffs in the game result from social communication between the players and depend on how well-connected the players are in the graph. Namely, when a player forms a link with another player it not only enables him to communicate with that player, but also with the players he is indirectly connected with through that player. Notice that the above description implies that we consider a model where link formation is one-sided, i.e., every link is initiated by exactly one player, while the benefits are multi-sided, i.e., possibly all players may use a link for communication. Taking a probabilistic choice approach, as explained above, means that we characterize players by their mixed strategies and that we identify a social communication network with a mixed-strategy profile of the players. In other words, a social communication network specifies for every possible graph the probability that it will actually be realized.

To model the evolution of social communication networks we appeal to the stochastic processes for reinforcement learning of Bush and Mosteller [6]. Players in these learning models adjust their mixed strategies over time in response to their experiences. The attractiveness of models for reinforcement learning follows from the characteristic that players respond to very limited information. Namely, experiences only consist of their own pure strategy and the payoff they receive. This characteristic fits with the natural assumption that a player's mixed strategy can not be observed by the other players. Furthermore, in models of reinforcement learning payoffs are used as parametrizations of players' responses to their experiences and should therefore not be interpreted as von Neumann-Morgenstern utilities. We focus on the process of habit formation as previously considered by Cross [8]. This process is referred to as a model of habit formation to denote that all experiences are positively reinforced, meaning that all experiences induce a player to increase the probability of the strategy just chosen. Payoffs only matter in that they determine how much the probability increases, i.e., payoffs determine the extent to which habit formation occurs.

Börgers and Sarin [5] show that at every finite point in time, assuming frequent communication and slow adjustment, the process of habit formation as considered by Cross [8] can be approximated by the well-known continuous time replicator dynamics. We use this result to select among the possible social communication networks. In particular, we specify an asymptotically stable set of social communication networks which consists of Nash equilibria of the corresponding game. Note that Nash equilibria are necessarily steady states of the replicator dynamics. When we restrict ourselves to the 3-player case we can show that only the two efficient 'cyclic' social communication networks are asymptotically stable. A social communication network is efficient if it maximizes the summation of the players' expected payoffs.

As mentioned before the model presented in this paper is a probabilistic choice approach to social communication networks. Hence, players tend to change the probabilities of forming links over time in directions that appear to be beneficial. For algebraic choice approaches to the evolution of social communication networks we refer to, e.g., Bala and Goyal [2] and Watts [19]. These deterministic models are based on the symmetric connections model as introduced by Jackson and Wolinsky [12]. In the symmetric connections model agents communicate with all other agents they are directly or indirectly connected with. Furthermore, the value of the communication depends on the number of links involved in the shortest path that connects a pair of agents. Both Bala and Goyal [2] and Watts [19] identify a social communication network with a graph and assume that players myopically form and sever links. Watts [19] shows convergence to an efficient network in case maintenance costs of links are small and closer connections are valued more than distant connections. Bala and Goyal [2] also find convergence to efficient networks in a variety of cases. Their model differs from Watts [19] as an agent can form a link without consent of other agent involved. Obviously, the costs of forming a link are then incurred by the agent who initiated the link.

The paper is organized as follows. In Section 2 we introduce the model. The related results of Börgers and Sarin [5] are stated in Section 3. Section 4 deals with the social communication game in case there are 3 players. The general n-player model is analyzed in Section 5. Finally, Section 6 concludes.

2 Model

Consider a finite normal-form game represented by a tuple $G = \langle N, (S_i)_{i \in N}, (\pi_i)_{i \in N} \rangle$, where $N = \{1, ..., n\}$ is a finite set of players. Every player $i \in N$ has a finite set of pure strategies, henceforth actions, given by

$$S_i := \{s_{ij} \mid j \in N \backslash \{i\}\}, \tag{1}$$

where action s_{ij} means that player i establishes a (direct) link with player $j \neq i$. From (1) it follows directly that $|S_i| = n - 1$ for all players $i \in N$. Since we identify establishing a link with an action, every player initiates exactly one link with another player when the game is played. For this reason we abstract from cost considerations of links. Instead of s_{ij} we will also use s_i to denote an element of S_i in case the identity of the player that player i establishes a link with can be neglected. The joint action set of the players is denoted by $S = \prod_{i \in N} S_i$ and we write $S_{-i} = \prod_{j \in N \setminus \{i\}} S_j$ for notational convenience. Furthermore, for an action tuple $s = (s_1, ..., s_n) \in S$ and a player $i \in N$ we let $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_{-i}$ and, with a slight abuse of notation, $s = (s_i, s_{-i}) \in S$. We denote the set of mixed strategies, henceforth strategies, for player i by

$$\Delta_i := \{\sigma_i : S_i \to \mathbb{R} \mid \forall s_i \in S_i : \sigma_i(s_i) \geq 0, \sum_{s_i \in S_i} \sigma_i(s_i) = 1\},$$

where $\sigma_i(s_i)$ is the probability that player i plays action s_i. In case the players' identity is important we use $\sigma_{ij} := \sigma_i(s_{ij})$ to denote the probability that player i establishes a link with player j. Analogous to the action case we use notations Δ, Δ_{-i}, and $\sigma = (\sigma_i, \sigma_{-i})$. By

$$\text{int}(\Delta_i) := \{\sigma_i : S_i \to \mathbb{R} \mid \forall s_i \in S_i : \sigma_i(s_i) > 0, \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

we denote the set of strategies for player i, such that a positive probability is assigned to all actions. The definitions of $\text{int}(\Delta)$ and $\text{int}(\Delta_{-i})$ are straightforward. For a strategy profile $\sigma_{-i} \in \Delta_{-i}$, we write $\sigma_{-i}(s_{-i}) := \prod_{j \in N \setminus \{i\}} \sigma_j(s_j)$ to denote the probability that the opponents of player i play the action profile $s_{-i} \in S_{-i}$.

In describing players' behavior we take a probabilistic choice approach as is conventional in psychology (see, e.g., Bush and Mosteller [6], Cross [9], and Estes [11]) and as is supported by experimental evidence (see, e.g., Suppes and Atkinson [18]). A probabilistic choice approach states that players' behavior is random, meaning that every player i does play an action $s_i \in S_i$, but this action is drawn from the probability distribution over his action set induced by his strategy σ_i. We therefore characterize every player i by a strategy $\sigma_i \in \Delta_i$. A possible interpretation of a player's strategy is as follows. Every player has to decide how to divide his available time (or effort) over his relations with the other $n - 1$ players. The more time a player spends on his relation with another player, the more likely it is that a link will actually be established. In fact, we will normalize total available time at 1 for all players, meaning that time can actually be interpreted as the probability of establishing a link. We will refer to a strategy profile σ as a social communication network.

The payoffs in the game are based on a modification of the symmetric connections model as introduced by Jackson and Wolinsky [12]. An outcome of the game s can be represented by a graph: the n players are the nodes and the links are the edges of the graph. As this is a model of social communication the players exchange information. Players directly communicate with those to whom they are directly linked. Through these links they also benefit from indirect communication with those to whom their adjacent nodes are linked, and so on. Consequently, the payoff a player receives from an outcome of the game is not only determined by the direct links but also the indirect ones. The payoff of communication obtained from other nodes depends on the distance to these nodes. In fact, if the minimal path in the graph that connects player i with another player j consists of l_{ij} links, then player i receives a payoff of $\delta^{l_{ij}}$ from his communication with player j. We assume that $0 < \delta < \frac{1}{n-1}$, meaning that payoffs decrease as the path connecting the players increases. An economic interpretation for this assumption is that information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Formally, the payoff player i associates with an outcome of the game s is equal to

$$\pi_i(s) := \sum_{j \neq i} \delta^{l_{ij}},$$

where $l_{ij} := l_{ij}(s)$ is the number of links in the shortest path between players i and j in the graph representing the outcome of the game s (setting $l_{ij} = \infty$ in case there is no path connecting player i and j). For later convenience we also introduce the expected payoff that player i associates with a social communication network σ:

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{i=1}^{n} \sigma_i(s_i) \right) \cdot \pi_i(s).$$

From the above description it follows that we consider a model where link formation is one-sided, i.e., every link is initiated by exactly one player, while benefits are multi-sided, i.e., possibly all players may use a link for communication. We refer to the game introduced above as the social communication game.

The n players maintain mutual relations and communicate repeatedly. The iterations of the social communication game are indexed by $k \in \mathbb{N} \cup \{0\}$. At each time $k \in \mathbb{N} \cup \{0\}$, every player $i \in N$ will be characterized by a strategy $\sigma_i^k \in \Delta_i$. By $\sigma_i^k(s_i)$ and $\sigma_{ij}^k := \sigma_i^k(s_{ij})$ we denote the probability that player i, at time k, plays action s_i and s_{ij}, respectively. The state of the game $\sigma^k = \left(\sigma_1^k, ..., \sigma_n^k\right) \in \Delta$ at time $k \in \mathbb{N} \cup \{0\}$ identifies a strategy for each player i, with the initial state of the game $\sigma^0 = (\sigma_1^0, ..., \sigma_n^0)$ exogenously given.

In this paper we consider an adjustment process that assumes all experiences to be habit forming, meaning that they make the player more likely to establish the same link again. How much more likely, however, depends on the payoff a player receives from the outcome of the game. Consider a fixed period $k \in \mathbb{N} \cup \{0\}$. In case player i realized a direct link with player j, habit formation gives rise the following adjustment of the social communication network σ^k:

$$\sigma_{ih}^{k+1} = \begin{cases} \left(1 - \sum_{h \neq i} \delta^{l_{ih}^k}\right) \cdot \sigma_{ih}^k + \left(\sum_{h \neq i} \delta^{l_{ih}^k}\right) \cdot 1 & \text{, if } h = j. \\ \left(1 - \sum_{h \neq i} \delta^{l_{ih}^k}\right) \cdot \sigma_{ih}^k & \text{, otherwise,} \end{cases} \tag{2}$$

where $l_{ij}^k := l_{ij}^k(s)$ is the number of links in the shortest path between players i and j in the graph representing the outcome of the game s at time k. Note that $\sum_{h \neq i} \delta^{l_{ih}^k}$ can be used as a weight because $0 < \delta < \frac{1}{n-1}$ implies that $0 < \sum_{h \neq i} \delta^{l_{ih}^k} < 1$. Given an initial state σ^0 the stochastic process (2) defines a Markov process $\left\{\sigma^k\right\}_{k \in \mathbf{N} \cup \{0\}}$ with infinite state space Δ.

3 Habit Formation and Replicator Dynamics

In this section we briefly discuss two results of Börgers and Sarin [5] who recently analyzed the habit-formation model of Cross [8] for general normal-form games. In Section 4 and Section 5 we will use their results to analyze the evolution of social communication networks.

Proposition 1 deals with the asymptotic behavior ($k \to \infty$) of the stochastic process (2). In fact, the result states that a player's asymptotic behavior consists of playing an action, i.e., a pure strategy. Hence, players will not divide their available time (or effort) over different relations, but instead focus on communicating with exactly one other player.

Proposition 1 [Proposition 2 and Remark 3 of Börgers and Sarin [5]] *For all initial states σ^0 the sequence $\left\{\sigma^k\right\}_{k \in \mathbf{N} \cup \{0\}}$ converges with probability 1, and its limit is an element of the joint action set S. Moreover, in case $\sigma^0 \in int(\Delta)$, every element of S has a positive probability of being the limit of $\left\{\sigma^k\right\}_{k \in \mathbf{N} \cup \{0\}}$.*

Note that Proposition 1 does not exclude the possibility of two different players, say player i and j, forming two mutual links, i.e., playing action s_{ij} and s_{ji}, respectively. Since a second mutual link is redundant for successful communication and limits the ability to communicate with others, efficiency of the resulting social communication network can not be guaranteed.

The proposition above deals with the asymptotic behavior of the stochastic process in discrete time. Alternatively, one could consider the behavior of the habit-formation process at finite points in time. To make the analysis tractable we focus on frequent communication, i.e., we analyze what happens when the time interval between successive rounds of communication goes to zero. Let the time that passes between two successive rounds of communication be denoted by η with $0 < \eta \leq 1$. Under the assumption that the rate at which the players adjust their strategy slows down at the same rate at which the time interval shrinks, the evolution of the social communication network is described by the following modification of (2) for all $i \in N$ and

$k \in \mathbb{N} \cup \{0\}$:

$$\sigma_{ih}^{\eta,k+1} = \begin{cases} \left(1 - \eta \cdot \sum_{h \neq i} \delta^{l_{ih}^{\eta,k}}\right) \cdot \sigma_{ih}^{\eta,k} + \left(\eta \cdot \sum_{h \neq i} \delta^{l_{ih}^{\eta,k}}\right) \cdot 1 & \text{, if } h = j, \\ \left(1 - \eta \cdot \sum_{h \neq i} \delta^{l_{ih}^{\eta,k}}\right) \cdot \sigma_{ih}^{\eta,k} & \text{, otherwise.} \end{cases} \tag{3}$$

Note that $\sigma^{\eta,k}$ is the state of the process at 'real' time ηk. The stochastic process (3) can be used to determine what happens to the evolution of the social communication network as $\eta \to 0$. We refer to this limit as the frequent communication case. Börgers and Sarin [5] show that under the assumptions of frequent communication and slow adjustment, actual adjustment of the habit-formation process can be approximated by expected adjustment at every finite time period. Furthermore, expected adjustment of the habit-formation process coincides with the well-known replicator dynamics in continuous time, i.e., a system of deterministic differential equations. This result is stated in Proposition 2.

Proposition 2 [Proposition 1 of Börgers and Sarin [5]] *Suppose that for all* $0 < \eta \leq 1$ *it holds that* $\sigma^{\eta,0} = \hat{\sigma}^0$ *with probability* 1*. Consider some* t *with* $0 \leq t < \infty$ *and let* $\eta \to 0$ *and* $\eta k \to t$*. Let* $\hat{\sigma}$ *be the solution of the continuous time replicator dynamics for initial value* $\hat{\sigma}^0$*. Then* $\sigma^{\eta,k}$ *converges in probability to* $\hat{\sigma}^t$*.*

We will use Proposition 2 to analyze the evolution of social communication networks. In fact, the above proposition implies that we can, at least to some extent, analyze the process of habit formation in finite time by looking at the steady states, and their stability properties, of the continuous time replicator dynamics that corresponds to the social communication game. As will be shown below the results obtained in this way differ significantly from the results as stated in Proposition 1. Furthermore, Proposition 2 implies that results from the huge literature on the replicator dynamics (see, e.g., Weibull [20] for a survey) can be applied.

4 The 3-Player Case

In this section we analyze the evolution of social communication networks with 3 players in finite time. In fact, there are two reasons for paying attention to this specific case. First, the 3-player case enables us to develop some intuition for the general n-player model. Second, and probably more important, the 3-player case gives somewhat stronger results than the n-player model.

According to Proposition 2 the evolution of social communication networks in finite time can be approximated (assuming frequent communication and slow adjustment) by looking at the expected adjustment of the stochastic process (2) or, equivalently, the continuous time replicator dynamics corresponding to the social communication game. The payoff matrix of the 3-player social communication game is represented in Figure 1. Note that with 3 players only 2^3 different outcomes of the

	s_{21}	s_{23}
s_{12}	$2\delta, \delta+\delta^2, \delta+\delta^2$	$2\delta, 2\delta, 2\delta$
s_{13}	$2\delta, \delta+\delta^2, \delta+\delta^2$	$\delta+\delta^2, \delta+\delta^2, 2\delta$

s_{31}

	s_{21}	s_{23}
s_{12}	$\delta+\delta^2, 2\delta, \delta+\delta^2$	$\delta+\delta^2, 2\delta, \delta+\delta^2$
s_{13}	$2\delta, 2\delta, 2\delta$	$\delta+\delta^2, \delta+\delta^2, 2\delta$

s_{32}

Figure 1: The 3-Player Social Communication Game

game, i.e., graphs, are possible since $|S_i| = 2$ for $i = 1, 2, 3$.

In the 3-player case the expected adjustment of the stochastic process (2) in period $k \in \mathbb{N} \cup \{0\}$, conditional on the state of the process in that period, is given by

$$\begin{aligned} & E\left[\sigma_{ij}^{k+1} - \sigma_{ij}^{k} \mid \sigma^k = \sigma\right] \\ = \; & \sigma_{ij}\left\{\left[1 - E\left[\Sigma_{h\neq i}\delta^{l_{ih}^k} \mid \sigma^k = \sigma, \sigma_{ij} = 1\right]\right] \cdot \sigma_{ij}\right. \\ & \left. + E\left[\Sigma_{h\neq i}\delta^{l_{ih}^k} \mid \sigma^k = \sigma, \sigma_{ij} = 1\right] \cdot 1 - \sigma_{ij}\right\} \\ & + (1-\sigma_{ij})\left\{\left[1 - E\left[\Sigma_{h\neq i}\delta^{l_{ih}^k} \mid \sigma^k = \sigma, \sigma_{ij} = 0\right]\right] \cdot \sigma_{ij} - \sigma_{ij}\right\} \end{aligned} \tag{4}$$

for all $k \in \mathbb{N} \cup \{0\}$, $i, j \in \{1, 2, 3\}$, and $i \neq j$. Using the payoff matrix of the social communication game as represented in Figure 1, the expected adjustment of the stochastic process (2) can easily be determined. Consider player $i = 1$. Straightforward calculations show that player 1's expected payoff of the social communication game in period k, conditional on the state of the process, is equal to

$$\begin{aligned} & E\left[\Sigma_{h\neq 1}\delta^{l_{1h}^k} \mid \sigma^k = \sigma\right] \\ = \; & 2\delta\left(\sigma_{12}\cdot\sigma_{23}\cdot\sigma_{31}\right) + 2\delta\left(\sigma_{12}\cdot\sigma_{21}\cdot\sigma_{31}\right) \\ & + \left(\delta+\delta^2\right)\left(\sigma_{12}\cdot\sigma_{23}\cdot\sigma_{32}\right) + \left(\delta+\delta^2\right)\left(\sigma_{12}\cdot\sigma_{21}\cdot\sigma_{32}\right) \\ & + 2\delta\left(\sigma_{13}\cdot\sigma_{21}\cdot\sigma_{32}\right) + 2\delta\left(\sigma_{13}\cdot\sigma_{21}\cdot\sigma_{31}\right) \\ & + \left(\delta+\delta^2\right)\left(\sigma_{13}\cdot\sigma_{23}\cdot\sigma_{32}\right) + \left(\delta+\delta^2\right)\left(\sigma_{13}\cdot\sigma_{23}\cdot\sigma_{31}\right) \\ = \; & 2\delta\left(\sigma_{12}\cdot\sigma_{23}\cdot\sigma_{31} + \sigma_{21}\cdot\sigma_{31} + \sigma_{13}\cdot\sigma_{21}\cdot\sigma_{32}\right) \\ & + \left(\delta+\delta^2\right)\left(\sigma_{12}\cdot\sigma_{21}\cdot\sigma_{32} + \sigma_{23}\cdot\sigma_{32} + \sigma_{13}\cdot\sigma_{23}\cdot\sigma_{31}\right). \end{aligned}$$

Consequently, player 1's conditional expected payoffs when $\sigma_{12} = 1$ and $\sigma_{12} = 0$, i.e., conditional on his own action, are given by

$$\begin{aligned} & E\left[\Sigma_{h \neq 1} \delta^{l_{1h}^k} \mid \sigma^k = \sigma, \sigma_{12} = 1\right] \\ = \; & 2\delta\left(\sigma_{23} \cdot \sigma_{31} + \sigma_{21} \cdot \sigma_{31}\right) + \left(\delta + \delta^2\right)\left(\sigma_{21} \cdot \sigma_{32} + \sigma_{23} \cdot \sigma_{32}\right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} & E\left[\Sigma_{h \neq 1} \delta^{l_{1h}^k} \mid \sigma^k = \sigma, \sigma_{12} = 0\right] \\ = \; & 2\delta\left(\sigma_{21} \cdot \sigma_{31} + \sigma_{21} \cdot \sigma_{32}\right) + \left(\delta + \delta^2\right)\left(\sigma_{23} \cdot \sigma_{32} + \sigma_{23} \cdot \sigma_{31}\right), \end{aligned} \tag{6}$$

respectively. Subsequently substituting (5) and (6) in (4) and rearranging terms shows that the expected adjustment of player 1's strategy in period k, conditional on the state of the process in that period, is given by

$$E\left[\sigma_{12}^{k+1} - \sigma_{12}^k \mid \sigma^k = \sigma\right] = \left(\delta - \delta^2\right)\sigma_{12}\left(1 - \sigma_{12}\right)\left(\sigma_{31} - \sigma_{21}\right) \tag{7}$$

and

$$E\left[\sigma_{13}^{k+1} - \sigma_{13}^k \mid \sigma^k = \sigma\right] = \left(\delta - \delta^2\right)\sigma_{13}\left(1 - \sigma_{13}\right)\left(\sigma_{21} - \sigma_{31}\right). \tag{8}$$

In fact, one of the equations (7) and (8) is redundant since $\sigma_{12}^k + \sigma_{13}^k = 1$ for all $k \in \mathbb{N} \cup \{0\}$. The conditional expected adjustment of the strategies of players 2 and 3 can be derived in a similar way.

As mentioned before, the conditional expected adjustment of the stochastic process (2) is equal to the continuous time replicator dynamics corresponding to the social communication game. Straightforward calculations show that the continuous time replicator dynamics in the 3-player social communication game is equal to

$$\dot{\sigma}_{ij} = \left(\delta - \delta^2\right)\sigma_{ij}\left(1 - \sigma_{ij}\right)\left(\sigma_{hi} - \sigma_{ji}\right) \tag{9}$$

for all $h, i, j \in \{1, 2, 3\}$, $h \neq i$, $h \neq j$, and $i \neq j$. Note that for every player i one equation in (9) is redundant, meaning that in the 3-player case the evolution of the social communication network can be described completely by a 3-dimensional dynamical system.

Below we analyze social communication networks in finite time by looking at the steady states, and their stability properties, of the continuous time replicator dynamics. In fact, the analysis is carried out in terms of the variables σ_{12}, σ_{23}, and σ_{31}. Proposition 3 gives the steady states of the replicator dynamics (9).

Proposition 3 *The set of steady states of the replicator dynamics (9) consists of the three isolated points* $(\sigma_{12}, \sigma_{23}, \sigma_{31}) = (0, 0, 0)$, $(\sigma_{12}, \sigma_{23}, \sigma_{31}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, *and* $(\sigma_{12}, \sigma_{23}, \sigma_{31}) = (1, 1, 1)$, *and the six edges of the unitcube* C, *i.e.,*

$$C = \left\{(\sigma_{12}, \sigma_{23}, \sigma_{31}) \in \mathbb{R}^3 \mid 0 \leq \sigma_{12} \leq 1, 0 \leq \sigma_{23} \leq 1, 0 \leq \sigma_{31} \leq 1\right\},$$

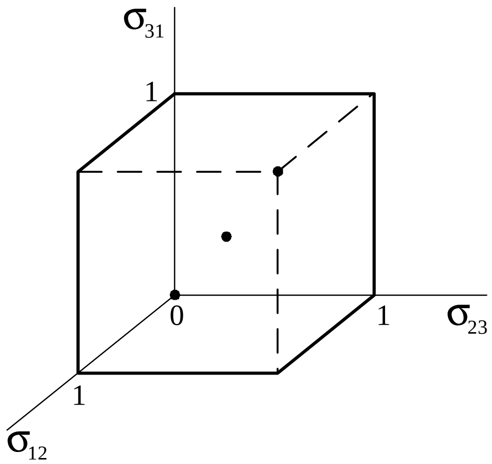

Figure 2: Steady states of the replicator dynamics with 3 players.

that have no point in common with $(0,0,0)$ *or* $(1,1,1)$.

The proof is straightforward and therefore omitted. Figure 2 illustrates the set of steady states. In general (see, e.g., Proposition 5.9 in Chapter 5 of Weibull [20]), Nash equilibria of a game are steady states of the corresponding replicator dynamics. The converse is also true for interior steady states. In the social communication game the boundary steady states $(0,0,0)$ and $(1,1,1)$ of the replicator dynamics are Nash equilibria of the social communication game. This follows directly from the payoff matrix as represented in Figure 1.

To select among the infinite number of steady states we consider their stability properties. Proposition 4 states that only two steady states are asymptotically stable, while all others are unstable.

Proposition 4 *The two steady states* $(\sigma_{12},\sigma_{23},\sigma_{31}) = (0,0,0)$ *and* $(\sigma_{12},\sigma_{23},\sigma_{31}) = (1,1,1)$ *of the replicator dynamics (9) are asymptotically stable with basins of attraction given by*

$$D_0 = \left\{(\sigma_{12},\sigma_{23},\sigma_{31}) \in \mathbb{R}^3 \mid \sigma_{31} < 1-\sigma_{23}, \sigma_{12} < 1-\sigma_{31}, \sigma_{23} < 1-\sigma_{12}\right\}$$

and

$$D_1 = \left\{(\sigma_{12},\sigma_{23},\sigma_{31}) \in \mathbb{R}^3 \mid \sigma_{31} > 1-\sigma_{23}, \sigma_{12} > 1-\sigma_{31}, \sigma_{23} > 1-\sigma_{12}\right\},$$

respectively. All other steady states of the replicator dynamics (9) are unstable.

Proof. The first part of the proof is based on Lyapunov's Direct Method. Consider the steady state $(\sigma_{12},\sigma_{23},\sigma_{31}) = (0,0,0)$. Let

$$A = \left\{(\sigma_{12},\sigma_{23},\sigma_{31}) \in \mathbb{R}^3 \mid \sigma_{12} = \sigma_{23} = \sigma_{31} = 0\right\},$$

the neighborhood $D = D_0$, and $v(\sigma_{12}, \sigma_{23}, \sigma_{31}) = \sigma_{12} + \sigma_{23} + \sigma_{31}$. Applying Theorem 6.4 in Chapter 6 of Weibull [20] immediately gives the required result. Now consider the steady state $(\sigma_{12}, \sigma_{23}, \sigma_{31}) = (1, 1, 1)$. Let

$$A = \left\{ (\sigma_{12}, \sigma_{23}, \sigma_{31}) \in \mathbb{R}^3 \mid \sigma_{12} = \sigma_{23} = \sigma_{31} = 1 \right\},$$

$D = D_1$, and $v(\sigma_{12}, \sigma_{23}, \sigma_{31}) = 3 - (\sigma_{12} + \sigma_{23} + \sigma_{31})$. Again, the result follows from applying Theorem 6.4 in Chapter 6 of Weibull [20].

To show that all other steady states are unstable we use the well-known eigenvalue analysis. The columns of the Jacobian matrix $J = [J_1, J_2, J_3]$ corresponding to the replicator dynamics (9) can easily seen to be given by

$$J_1 = \left(\delta - \delta^2\right) \left[(\sigma_{31} + \sigma_{23} - 1)(1 - 2\sigma_{12}), \sigma_{23}(1 - \sigma_{23}), \sigma_{31}(1 - \sigma_{31}) \right]',$$

$$J_2 = \left(\delta - \delta^2\right) \left[\sigma_{12}(1 - \sigma_{12}), (\sigma_{12} + \sigma_{31} - 1)(1 - 2\sigma_{23}), \sigma_{31}(1 - \sigma_{31}) \right]',$$

and

$$J_3 = \left(\delta - \delta^2\right) \left[\sigma_{12}(1 - \sigma_{12}), \sigma_{23}(1 - \sigma_{23}), (\sigma_{23} + \sigma_{12} - 1)(1 - 2\sigma_{31}) \right]'.$$

Stability properties of a steady state can be deduced from the Jacobian matrix by evaluating the matrix at the steady state and calculating the eigenvalues. Since $\left(\delta - \delta^2\right) > 0$ it follows that in all steady states unequal to $(0, 0, 0)$ or $(1, 1, 1)$ there is always at least one strictly positive eigenvalue, which implies that the steady states are unstable. ■

The asymptotic stability of the two 'cyclic' social communication networks given by $(\sigma_{12}, \sigma_{23}, \sigma_{31}) = (0, 0, 0)$ and $(\sigma_{12}, \sigma_{23}, \sigma_{31}) = (1, 1, 1)$ can be explained by the fact that these are the only 3-player social communication networks for which there is no positive probability associated with the event of a 'double link', i.e., a pair of players being connected by means of two mutual links. As the presence of one link is already sufficient for successful communication, adding a second link is redundant and only decreases the possibility to communicate with other players, which in turn results in a lower payoff from the social communication network. For the same reason, the two 'cyclic' social communication networks are also the only efficient social communication networks. Recall that a social communication network is efficient if it maximizes the summation of the players' expected payoffs.

Finally, note that the first part of Proposition 4 is consistent with Proposition 2 in Ritzberger and Weibull [17]. Namely, the two 'cyclic' social communication networks are strict Nash equilibria of the social communication game according to the payoff matrix as represented in Figure 1. Ritzberger and Weibull [17] show that strict Nash equilibria coincide with asymptotically stable steady states in the continuous time replicator dynamics.

5 The n-Player Case

Consider the model as introduced in Section 2. In this section we will specify an asymptotically stable set of social communication networks which consists of Nash equilibria of the corresponding social communication game. Recall that Nash equilibria are necessarily steady states of the continuous time replicator dynamics (see, e.g., Proposition 5.9 in Chapter 5 of Weibull [20]).

In order to prove existence of an asymptotically stable set of social communication networks we introduce the following notation. For every nonempty subset $H_i^c \subset S_i$ of player i's actions, let $\Delta_i(H_i^c) \subset \Delta_i$ be the face of the simplex Δ_i spanned by H_i^c, i.e.,

$$\Delta_i(H_i^c) = \{\sigma_i \in \Delta_i \mid C(\sigma_i) \subset H_i^c\},$$

where $C(\sigma_i) = \{s_{ij} \in S_i \mid \sigma_i(s_{ij}) > 0\}$ is the support (or carrier) of $\sigma_i \in \Delta_i$. Likewise, for every collection of nonempty subsets $H_i^c \subset S_i$ of actions, one for every player $i \in N$, let $H^c = \times_{i \in N} H_i^c \subset S$, and let the closed and convex set $\Delta(H^c)$ be the face of the set of strategy profiles Δ spanned by H^c, i.e., $\Delta(H^c) = \times_{i \in N} \Delta_i(H_i^c)$. Furthermore, let

$$\Phi = \left\{\sigma \in \Delta^{NE} \mid \sigma_i(s_{ij}) \cdot \sigma_j(s_{ji}) = 0 \text{ for all } i, j \in N \text{ such that } i \neq j\right\},$$

where Δ^{NE} denotes the set of Nash equilibria of the social communication game. Below we will show that the set Φ is asymptotically stable in the continuous time replicator dynamics. It can easily be verified that the set Φ is the union of a finite (recall that this paper deals with finite normal-form games) number of faces $\Delta(H^c)$, with $c = 1, ..., C$, i.e., $\Phi = \cup_{c=1,...,C} \Delta(H^c)$. In fact, we refer to $\Delta(H^c)$, $c = 1, ..., C$, as the components of Φ.

The proof of asymptotic stability of the set Φ is conducted through the following lemma. The basic idea underlying the result in Lemma 5 is that the only way of leaving a component of Φ without being confronted with a decreasing payoff is by entering one of the other components.

Lemma 5 *There exists some open set $U \supset \Phi$ such that*

$$[\sigma \in U \cap \Delta \text{ and } u_i(s_i, \sigma_{-i}) \geq u_i(\sigma)] \Rightarrow (s_i, \sigma_{-i}) \in \Phi.$$

Proof. Consider a component $\Delta(H^c)$ of Φ. It will be shown first that if there exists some $i \in N$ such that $s_i \notin H_i^c$ and $\gamma := (s_i, \sigma_{-i}) \notin \Phi$ for all $\sigma_{-i} \in \times_{j \neq i} \Delta_j\left(H_j^c\right)$, then $u_i(s_i, \sigma_{-i}) < u_i(\sigma)$ for all $\sigma \in \Delta(H^c)$. Suppose there exists a $s_i \notin H_i^c$ and $\sigma \in \Delta(H^c)$ such that $u_i(s_i, \sigma_{-i}) \geq u_i(\sigma)$. Since $\sigma \in \Delta(H^c) \subset \Phi$ this immediately implies that $\gamma \in \Delta^{NE}$. Furthermore, since $s_i \notin H_i^c$ and $\sigma \in \Delta(H^c)$ it also holds that

$\gamma_i(s_{ij}) \cdot \gamma_j(s_{ji}) = 0$ for all $i, j \in N$, with $i \neq j$. Consequently, $\gamma \in \Phi$, which gives the required contradiction.

Suppose that for some $i \in N$ it holds that $s_i \notin H_i^c$. Then by continuity of u_i and compactness of $\Delta(H^c)$, there exists an open set $U_{is_i}^c$ such that $\Delta(H^c) \subset U_{is_i}^c$ and $u_i(s_i, \sigma_{-i}) < u_i(\sigma)$ for all $\sigma \in \left(U_{is_i}^c \cap \Delta\right) \backslash \Phi$. Let $U_i^c = \cap_{s_i \notin H_i^c} U_{is_i}^c$. Doing likewise for all other players and taking the finite intersection $U^c = \cap_{i \in N} U_i^c$ of open sets all of which contain $\Delta(H^c)$, one obtains an open set U^c that contains $\Delta(H^c)$ and for which it holds that $u_i(s_i, \sigma_{-i}) < u_i(\sigma)$ for all $i \in N$, $s_i \notin H_i^c$, and $\sigma \in (U^c \cap \Delta) \backslash \Phi$. Once more taking the finite union $U = \cup_{c=1,\ldots,C} U^c$ gives an open set as specified in Lemma 5. ■

Now we are able to prove asymptotic stability of the set Φ.

Proposition 6 *The set $\Phi \subset \Delta$ is asymptotically stable in the replicator dynamics.*

Proof. Let U be as specified in Lemma 5. This implies that for every component $\Delta(H^c)$ there exists some $\bar{\varepsilon}^c$ such that for every $\varepsilon \in (0, \bar{\varepsilon}^c)$ and every player $i \in N$ the ε-slice,

$$B_i^c(\varepsilon) = \{\sigma_i \in \Delta_i \mid \sigma_{is_i} < \varepsilon \text{ for all } s_i \notin H_i^c\},$$

contains the corresponding face $\Delta_i(H_i^c)$ and $B^c(\varepsilon) = \times_{i \in N} B_i^c(\varepsilon)$ is contained in U^c. Obviously, $B(\varepsilon) = \cup_{c=1,\ldots,C} B^c(\varepsilon)$ is therefore contained in U. By continuity of u_i for all $i \in N$, we may assume that for any $\sigma \in B(\varepsilon) \cap \Delta$, $i \in N$, and $(s_i, \sigma_{-i}) \notin \Phi$, it holds that $u_i(s_i, \sigma_{-i}) - u_i(\sigma) < -\delta$ for some $\delta > 0$. Hence,

$$\dot{\sigma}_{is_i} = [u_i(s_i, \sigma_{-i}) - u_i(\sigma)]\, \sigma_{is_i} < -\delta \sigma_{is_i} \tag{10}$$

for any $\sigma \in B(\varepsilon) \cap \Delta$, $i \in N$, and $(s_i, \sigma_{-i}) \notin \Phi$.

Let ξ denote the solution mapping for the replicator dynamics, i.e., $\xi(t, \sigma^0) \subset \Delta$ is the popluation state at time $t \in \mathbb{R}$ if the initial state is $\sigma^0 \in \Delta$. Then (10) implies that $\xi_{is_i}(., \sigma^0) : \mathbb{R} \to \Delta$ decreases monotonically to zero for any $i \in N$, and $(s_i, \sigma_{-i}) \notin \Delta(H)$ from any initial state $\sigma^0 \in B(\varepsilon) \cap \Delta$. Thus Φ is asymptotically stable. ■

In Section 4 it was shown that in the 3-player case the two asymptotically stable social communication networks are also efficient. Recall that efficiency does not mean Pareto efficiency, but instead refers to the total sum of the players' expected payoffs being maximal. Figure 3 shows that in general efficiency of the social communication networks contained in the asymptotically stable set Φ can not be guaranteed. In fact, it can easily be verified that the 6-player social communication network represented in Figure 3(a) is an element of the set Φ, and that the summation of the players'

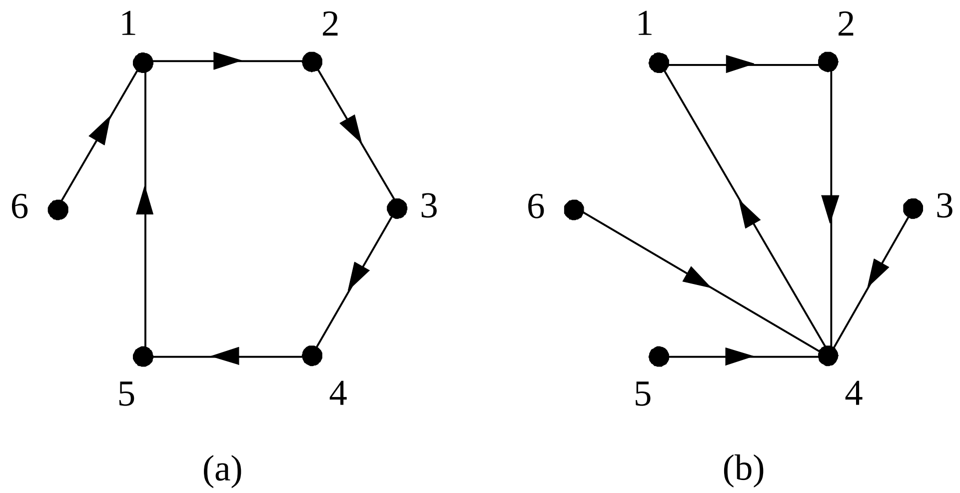

Figure 3: (a) A 6-player social communication network $\sigma \in \Phi$ which is not efficient and (b) an efficient 6-player social communication network $\sigma \in \Phi$.

payoffs equals $12\delta + 14\delta^2 + 4\delta^3$. Since the summation of the players' payoffs in the 6-player social communication network illustrated in Figure 3(b) is equal to $12\delta + 18\delta^2$, which is obviously larger than $12\delta + 14\delta^2 + 4\delta^3$, the social communication network in Figure 3(a) is not efficient. Note that the social communication network in Figure 3(b) is also an element of Φ. Finally, straightforward calculations show that efficiency is implied by asymptotic stability in case the number of players is at most 5.

6 Concluding Remarks

We have analyzed a social communication game, where the players' pure strategies consist of forming links with the other players. Taking a probabilistic choice approach we identified a social communication network with a mixed-strategy profile. The evolution of networks is modeled using a stochastic process of habit formation in discrete time. In finite time and assuming frequent communication and slow adjustment, the stochastic process can be approximated by the replicator dynamics. We were able to specify an asymptotically stable set of social communication networks which consists of Nash equilibria of the social communication game. Furthermore, in the 3-player case this set reduces to the set consisting of the two efficient 'cyclic' social communication networks.

Topics for future research include considering a social communication model with a stochastic process that involves an endogenous aspiration level (see, e.g., Börgers and Sarin [4]). Furthermore, it may be interesting to consider other payoff structures than the symmetric connections model.

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